

STRATIFYING SYSTEMS OVER THE HEREDITARY PATH ALGEBRA WITH QUIVER $\mathbb{A}_{p,q}$

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ABSTRACT. The authors have proved in [J. Algebra Appl. 14 (2015), no. 6] that the size of a stratifying system over a finite-dimensional hereditary path algebra A is at most n , where n is the number of isomorphism classes of simple A -modules. Moreover, if A is of Euclidean type a stratifying system over A has at most $n - 2$ regular modules. In this work, we construct a family of stratifying systems of size n with a maximal number of regular elements, over the hereditary path algebra with quiver $\mathbb{A}_{p,q}$, canonically oriented.

1. PRELIMINARIES

Throughout this article the term algebra means associative and finite dimensional basic algebra over an algebraically closed field K . Therefore any algebra can be viewed as a quotient $A = KQ/I$ of a path algebra KQ , where Q is a quiver and I is an admissible ideal of KQ . For a vertex v of Q , S_v will denote the simple module associated with v . The projective cover and the injective envelope of S_v will be denoted by P_v and I_v , respectively.

If A is a K -algebra then $\text{mod } A$ denotes the category whose objects are finitely generated right A -modules, $D : \text{mod } A \rightarrow \text{mod } A^{\text{op}}$ the usual duality $\text{Hom}_K(_, K)$, τ the Auslander-Reiten translation, and $\Gamma(\text{mod } A)$ the Auslander-Reiten quiver of $\text{mod } A$.

The concept of stratifying system (*s.s.*) was introduced as a generalization of the standard modules by Erdmann and Sáenz in [5]. Later Marcos et al. introduced in [6, 7] *s.s.* via relative simple modules and via relative projective modules. So there are, in the literature, various equivalent definitions of stratifying systems. In this work we use the following definition.

Definition 1. [6] *A stratifying system (*s.s.*) of size t consists of a completely ordered set $X = \{X_i\}_{i=1}^t$ of indecomposable A -modules satisfying the following conditions:*

- (1) $\text{Hom}_A(X_j, X_i) = 0$, for $j > i$.
- (2) $\text{Ext}_A^1(X_j, X_i) = 0$, for $j \geq i$.

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Let A be a hereditary K -algebra. An A -module M is called **exceptional** if $\text{End}_A(M) \cong K$ and $\text{Ext}_A^1(M, M) = 0$ and a sequence $X = (X_1, \dots, X_t)$ is called **exceptional** if it is a sequence of exceptional modules satisfying $\text{Hom}_A(X_i, X_j) = 0$, for $i > j$ and $\text{Ext}_A^1(X_i, X_j) = 0$ for $i \geq j$. A sequence $X = (X_1, \dots, X_n)$ is said to be **complete** if n is the number of isomorphism classes of simple A -modules.

Lemma 2. *If A is a hereditary K -algebra, then the sequence of A -modules (X_1, \dots, X_t) is an exceptional sequence if and only if, the set $X = \{X_i\}_{i=1}^t$ is a s.s. over A .*

Proof. The only fact that needs to be observed is that if an indecomposable module does not have self extensions then its endomorphism ring is K . This is a consequence of [1, Chapter VIII, Lemma 3.3], which states that if A is a finite dimensional hereditary algebra over an algebraically closed field K and T_1 and T_2 are two indecomposable A modules with $\text{Ext}_A^1(T_1, T_2) = 0$ then every nonzero morphism between T_1 and T_2 is a monomorphism or an epimorphism. In particular if $T = T_1 = T_2$ then all nonzero morphism is an isomorphism, so $\text{End}_A(T) \cong K$. \square

Since all the algebras in this work are hereditary, the former lemma gives a justification for identifying a s.s. $X = \{X_i\}_{i=1}^t$ with the exceptional sequence (X_1, \dots, X_t) .

We now stating some known statements, which we will use.

Lemma 3. [4, Lemma 1] *Any exceptional sequence $(X_1, \dots, X_a, Z_1, \dots, Z_c)$ can be enlarged to a complete sequence $(X_1, \dots, X_a, Y_1, \dots, Y_b, Z_1, \dots, Z_c)$.*

Lemma 4. [4, Lemma 2] *If (X_1, \dots, X_n) and (Y_1, \dots, Y_n) are complete exceptional sequences such that $X_j \cong Y_j$ for $j \neq i$, then $X_i \cong Y_i$.*

The authors of this work, proved in [2], Lemma 3.1, that if $A \cong KQ$ is a hereditary algebra then the size of a s.s. over A is at most $n = |Q_0|$. Moreover, if A is a hereditary algebra of Euclidean type and X is a s.s. of regular modules then the size of X is at most $n - 2$.

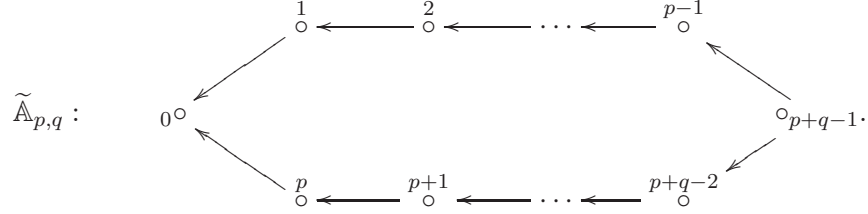
As in the case of exceptional sequences, we say that a s.s. X is complete (c.s.s.) if the size of X is n .

The algebra $K(\tilde{\mathbb{A}}_{p,q})$ will denote the hereditary algebra whose quiver is of type $\tilde{\mathbb{A}}_{p,q}$, canonically oriented, that is it has p consecutive arrows in one direction and q consecutive arrows in the other direction, we will always assume, without loss of generality, that it has p consecutive arrows counter-clockwise and q consecutive arrows clockwise.

The main objective of this article is to construct c.s.s. with maximal number of regular elements over $K(\tilde{\mathbb{A}}_{p,q})$.

2. THE ALGEBRA $K(\tilde{\mathbb{A}}_{p,q})$

In this section p and q are integers such that $1 \leq p \leq q$. Let $\tilde{\mathbb{A}}_{p,q}$ be the canonically oriented Euclidean quiver, whose picture is the following:



For any non-negative integers m, n we set by definition

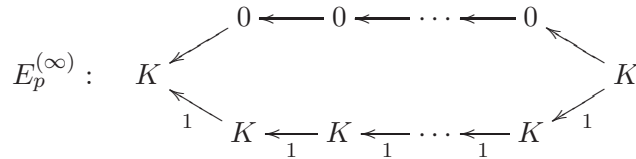
$$[m, n] := \begin{cases} \{m, m+1, \dots, n\} & \text{if } m \leq n \\ \emptyset, & \text{if } m > n. \end{cases}$$

We give a construction of s.s. over $A = K\tilde{\mathbb{A}}_{p,q}$ with a maximal number regular modules. The Auslander-Reiten quiver of such algebra, has a preprojective component, a preinjective component, an infinite family of homogeneous tubes, and 2 connected regular components, that are tubes but not homogeneous, which are denoted by \mathcal{T}_∞ and \mathcal{T}_0 (see Theorem 5).

Observe that if an indecomposable module T is in a homogeneous tube then $\tau(T) = T$, therefore it has self extension, that is $\text{Ext}_A^1(T, T) \neq 0$.

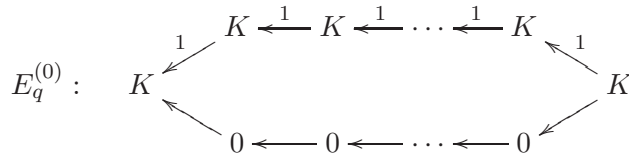
We give the description of the non-homogeneous tubes.

- (a) The simple regular representations in the tube \mathcal{T}_∞ are the simple modules $E_i^{(\infty)} = S_i, i \in [1, p-1]$, and



satisfying $\tau E_{i+1}^{(\infty)} = E_i^{(\infty)}$, for $i \in [1, p-1]$, and $\tau E_1^{(\infty)} = E_p^{(\infty)}$. (2.1)

- (b) The simple regular representations in the tube \mathcal{T}_0 are the simple modules $E_j^{(0)} = S_{p+j-1}, j \in [1, q-1]$ and



satisfying $\tau E_{j+1}^{(0)} = E_j^{(0)}$, for $j \in [1, q-1]$, and $\tau E_1^{(0)} = E_q^{(0)}$. (2.2)

- (c) The simple regular representation in the tube \mathcal{T}_λ , with $\lambda \in K \setminus \{0\}$ is

$$E^{(\lambda)} : \quad \begin{array}{ccccccc} & & K & \xleftarrow{1} & K & \xleftarrow{1} & \cdots & \xleftarrow{1} & K & & \\ & \swarrow \lambda & & & & & & & & \nwarrow 1 & \\ K & & & & & & & & & & K \\ & \nwarrow 1 & & & & & & & & \swarrow 1 & \\ & & K & \xleftarrow{1} & K & \xleftarrow{1} & \cdots & \xleftarrow{1} & K & & \end{array}$$

The following theorem give us information about the tubular components of $\Gamma(\text{mod } K\tilde{\mathbb{A}}_{p,q})$.

Theorem 5. [8, Chap. XIII, Theorem 2.5] *Let $A = K\tilde{\mathbb{A}}_{p,q}$. Then every component in the regular part $\mathcal{R}(A)$ of $\Gamma(\text{mod } A)$ is one of the following stable tubes:*

- (1) *The tube \mathcal{T}_∞ of rank p containing $E_1^{(\infty)}, \dots, E_p^{(\infty)}$,*
- (2) *The tube \mathcal{T}_0 of rank q containing $E_1^{(0)}, \dots, E_q^{(0)}$,*
- (3) *The tube \mathcal{T}_λ of rank 1 containing $E^{(\lambda)}$, with $\lambda \in K \setminus \{0\}$.*

Where $E_j^{(\infty)}$, $E_i^{(0)}$ and $E^{(\lambda)}$ are the simple regular A -modules previously defined.

We use the simple regular modules in the tubes \mathcal{T}_∞ and \mathcal{T}_0 to build a s.s. of size $p + q - 2$ over $K\tilde{\mathbb{A}}_{p,q}$. Let F_i denote $E_{p-i}^{(\infty)}$, for $i \in [1, p-1]$, and let G_i denote $E_{q-i}^{(0)}$, for $i \in [1, q-1]$. Then we have that

$$\tau F_i = F_{i+1}, \text{ for } i \in [1, p-2], \tau F_{p-1} = E_p^{(\infty)} \text{ and } \tau E_p = F_1, \quad (2.3)$$

$$\tau G_i = G_{i+1}, \text{ for } i \in [1, q-2], \tau E_q = G_1 \text{ and } \tau E_q = G_1. \quad (2.4)$$

We set, $\mathcal{F} := \{F_1, \dots, F_{p-1}\}$, $F := (F_1, \dots, F_{p-1})$, $\mathcal{G} := \{G_1, \dots, G_{q-1}\}$ and $G := (G_1, \dots, G_{q-1})$.

Remark 2.1. *We remark that \mathcal{F} is empty if $p = 1$, also \mathcal{G} is empty if $q = 1$. In the case when $p = 1$ and $q = 1$ our algebra is the Kronecker algebra whose stratifying systems where described in [2]. So we assume from now on that $q > 1$.*

Proposition 6. *Let \mathcal{F} and \mathcal{G} be the sets of indecomposable modules defined above. Then \mathcal{F} and \mathcal{G} are s.s., unless $p = 1$ in which case \mathcal{F} is empty.*

Proof. This is a consequence of (2.3), (2.4) and the Auslander-Reiten formula. \square

Since Lemma 2 allows us to identify a s.s. with an exceptional sequence, using the previous proposition we can see that F and G are s.s.

We denote by (F, G) the sequence where the elements of F appear before the elements of G . Note that as the elements of F and G are in different orthogonal tubes (see Theorem 5) then (F, G) is a s.s.

Let $\{S_i\}_{i=1}^n$ be a complete set of the isomorphism classes of simple A -modules. The support of an A -module M is the set

$$\text{supp } M := \{i \in [1, n] \mid [M : S_i] \neq 0\},$$

where $[M : S_i]$ is the number of composition factors of M that are isomorphic to S_i , and the support of a set of A -modules \mathcal{X} is

$$\text{Supp } \mathcal{X} := \cup_{M \in \mathcal{X}} \text{supp } M.$$

If the elements of \mathcal{X} are regular modules, $i \in \mathbb{Z} \setminus \{0\}$ we denote by $\tau^i(\mathcal{X})$ the set

$$\tau^i(\mathcal{X}) := \{\tau^i M \mid M \in \mathcal{X}\}.$$

Observing the behavior of the indecomposable modules in the standard tubes above, we state the following lemma whose proof we leave to the reader.

Lemma 7. *Let \mathcal{F} and \mathcal{G} be the sets of indecomposable modules defined above, with $p > 1$. Then*

- (1) (a) $\tau^i(\mathcal{F}) = (\mathcal{F} \setminus \{F_i\}) \cup \{E_p^{(\infty)}\}$, for $i \in [1, p-1]$.
- (b) $\tau^{-i}(\mathcal{F}) = (\mathcal{F} \setminus \{F_{p-i}\}) \cup \{E_p^{(\infty)}\}$, for $i \in [1, p-1]$.
- (c) $\tau^p(\mathcal{F}) = \tau^{-p}(\mathcal{F}) = \mathcal{F}$.
- (d) For each $n \in \mathbb{N}$, $n \neq 0$, we have that

$$\tau^n(\mathcal{F}) = \begin{cases} \mathcal{F}, & \text{if } p \mid n \\ \tau^r(\mathcal{F}), & \text{if } n \equiv r \pmod{p}, \text{ with } r \in [1, p-1]. \end{cases}$$

$$\tau^{-n}(\mathcal{F}) = \begin{cases} \mathcal{F}, & \text{if } p \mid n \\ \tau^{-r}(\mathcal{F}), & \text{if } n \equiv r \pmod{p}, \text{ with } r \in [1, p-1]. \end{cases}$$

- (2) (a) $\tau^i(\mathcal{G}) = (\mathcal{G} \setminus \{G_i\}) \cup \{E_q^{(0)}\}$, for $i \in [1, q-1]$.
- (b) $\tau^{-i}(\mathcal{G}) = (\mathcal{G} \setminus \{G_{q-i}\}) \cup \{E_q^{(0)}\}$, for $i \in [1, q-1]$.
- (c) $\tau^q(\mathcal{G}) = \tau^{-q}(\mathcal{G}) = \mathcal{G}$.
- (d) For each $n \in \mathbb{N} \setminus \{0\}$ we have that

$$\tau^n(\mathcal{G}) = \begin{cases} \mathcal{G}, & \text{if } q \mid n \\ \tau^r(\mathcal{G}), & \text{if } n \equiv r \pmod{q}, \text{ with } r \in [1, q-1] \end{cases}$$

$$\tau^{-n}(\mathcal{G}) = \begin{cases} \mathcal{G}, & \text{if } q \mid n \\ \tau^{-r}(\mathcal{G}), & \text{if } n \equiv r \pmod{q}, \text{ with } r \in [1, q-1] \end{cases}$$

The lemma above give us the following corollary.

Corollary 8. *Let $n, r, p \in \mathbb{N}, p > 1$ and $n \neq 0$. Then:*

- (1) (a) If $p \mid n$, then $\text{Supp } \tau^n \mathcal{F} = [1, p-1] = \text{Supp } \mathcal{F}$.
- (b) If $p \nmid n$ and $r \in [1, p-1]$ is such that $n \equiv r \pmod{p}$, then

$$\begin{aligned} \text{Supp } \tau^n \mathcal{F} = \tau^r \mathcal{F} &= [0, p+q-1] \setminus \{p-r\}, \\ \text{Supp } \tau^{-n} \mathcal{F} = \tau^{-r} \mathcal{F} &= [0, p+q-1] \setminus \{r\}. \end{aligned}$$
- (2) (a) If $q \mid n$, then $\text{Supp } \tau^n \mathcal{G} = [p, p+q-2] = \mathcal{G}$.

(b) If $q \nmid n$ and $r \in [1, q-1]$ is such that $n \equiv r \pmod{q}$, then

$$\begin{aligned} \text{Supp } \tau^n \mathcal{G} = \tau^r \mathcal{G} &= [0, p+q-1] \setminus \{p+q-r-1\}, \\ \text{Supp } \tau^{-n} \mathcal{G} = \tau^{-r} \mathcal{G} &= [0, p+q-1] \setminus \{p+r-1\}. \end{aligned}$$

The following lemma will be used.

Lemma 9. *Let A be a hereditary algebra. Then*

(1) *If P_j and P_m are projective indecomposable A -modules then*

$$\text{Ext}^1(\tau^{-t} P_j, \tau^{-t-r} P_m) = 0, \text{ for all } t, r \geq 0.$$

(2) *If I_j and I_m are injective indecomposable A -modules then*

$$\text{Ext}^1(\tau^t I_j, \tau^{t-r} I_m) = 0, \text{ for all } t \geq r \geq 0.$$

Proof. We just prove the statement (1). The proof of (2) is similar. By the Auslander-Reiten formula $\text{Ext}_A^1(\tau^{-t} P_j, \tau^{-t-r} P_m) \cong D \text{Hom}_A(\tau^{-r-1} P_m, P_j)$. But $\text{Hom}_A(\tau^{-r-1} P_m, P_j) = 0$, otherwise P_j would have a non-projective predecessor in $\Gamma(\text{mod } A)$. \square

3. STRATIFYING SYSTEMS OVER THE ALGEBRA $K(\tilde{\mathbb{A}}_{p,q})$

We denote by (F, G, Y) an exceptional sequence where the elements of F appear before the elements of G and these last ones before Y . We also use the same notation for the associated *s.s.*

Proposition 10. *If Y is a postprojective A -module such that (F, G, Y) is a *s.s.*, then Y is one of the modules in the following list:*

- (1) P_0 .
- (2) P_{p+q-1} .
- (3) $\tau^{-t} P_0$, with $t \geq 1$ such that $p \mid t$ and $q \mid t$.
- (4) $\tau^{-t} P_{p+q-1}$, with $t \geq 1$ such that $p \mid t$ and $q \mid t$.
- (5) $\tau^{-t} P_{p-r}$, with $t \geq 1$ such that $q \mid t$, $t \equiv r \pmod{p}$ and $r \in [1, p-1]$.
- (6) $\tau^{-t} P_{p+q-r-1}$, with $t \geq 1$ such that $p \mid t$, $t \equiv r \pmod{q}$ and $r \in [1, q-1]$.

Proof. If M is a regular module, then

$$\text{Ext}_A^1(\tau^{-t} P_j, M) \cong D \text{Hom}_A(\tau^{-1} M, \tau^{-t} P_j) = 0, \text{ for } t \geq 0.$$

Because of that, to prove that $(F, G, \tau^{-t} P_j)$ is a *s.s.* we need to show that

$$\text{Hom}_A(\tau^{-t} P_j, \mathcal{F}) = 0 = \text{Hom}_A(\tau^{-t} P_j, \mathcal{G}).$$

On the other hand, as a consequence of Auslander-Reiten formula, we have that

$$\text{Hom}_A(\tau^{-t} P_j, \mathcal{F}) \cong \text{Hom}_A(P_j, \tau^t \mathcal{F}) \text{ and } \text{Hom}_A(\tau^{-t} P_j, \mathcal{G}) \cong \text{Hom}_A(P_j, \tau^t \mathcal{G}).$$

It follows that $(F, G, \tau^{-t} P_j)$ is a *s.s.* if and only if $j \in (\text{Supp } \tau^t \mathcal{G})' \cap (\text{Supp } \tau^t \mathcal{F})'$, where $(\text{Supp } \tau^t \mathcal{G})'$ denotes the complement of the set $(\text{Supp } \tau^t \mathcal{G})$ with relation to the set $[0, p+q-1]$.

Case 1. If $t = 0$, then $Y \cong P_j$ then (F, G, P_j) is a *s.s.* if and only if $j \in (\text{Supp } \mathcal{F})' \cap (\text{Supp } \mathcal{G})' = \{0, p + q - 1\}$.

Case 2. If $p \mid t$ and $q \mid t$ then, according to Corollary 8, $\text{Supp } \tau^t \mathcal{F} = \text{Supp } \mathcal{F}$ and $\text{Supp } \tau^t \mathcal{G} = \text{Supp } \mathcal{G}$. Therefore, $(F, G, \tau^{-t} P_j)$ is a *s.s.* if and only if $j \in \{0, p + q - 1\}$.

Case 3. If $q \mid t$ and $t \equiv r \pmod{p}$, with $r \in [1, p - 1]$, then by Corollary 8 we have that $(\text{Supp } \tau^t \mathcal{F})' = \{p - r\}$ and $(\text{Supp } \tau^t \mathcal{G})' = [0, p - 1] \cup \{p + q - 1\}$. Therefore $(F, G, \tau^{-t} P_j)$ is a *s.s.* if and only if $j = p - r$.

Case 4. If $p \mid t$ and $q \nmid t$, then similarly to the previous case we conclude that $(F, G, \tau^{-t} P_j)$ is a *s.s.* if and only if $j = p + q - r - 1$, where $t \equiv r \pmod{q}$, $r \in [1, q - 1]$.

Case 5. If $t \equiv r_1 \pmod{p}$ and $t \equiv r_2 \pmod{q}$, with $r_1 \in [1, p - 1]$ and $r_2 \in [1, q - 1]$ then

$$(\text{Supp } \tau^{r_1} \mathcal{F})' \cap (\text{Supp } \tau^{r_2} \mathcal{G})' = \{p - r_1\} \cap \{p + q - r_2 - 1\} = \emptyset.$$

□

Proposition 11. *If Y is a preinjective A -module such that (F, G, Y) is a *s.s.*, then Y is one of the modules in the following list*

- (1) $\tau^t I_p$, with $t \geq 1$ such that $t \equiv p - 1 \pmod{p}$ and $q \mid t$.
- (2) $\tau^t I_1$, with $t \geq 1$ such that $t \equiv q - 1 \pmod{q}$ and $p \mid t$.
- (3) $\tau^t I_0$, with $t \geq 1$ such that $t \equiv p - 1 \pmod{p}$ and $t \equiv q - 1 \pmod{q}$.
- (4) $\tau^t I_{p+q-1}$, with $t \geq 1$ such that $t \equiv p - 1 \pmod{p}$ and $t \equiv q - 1 \pmod{q}$.
- (5) $\tau^t I_{r+1}$, with $t \geq 1$ such that $t \equiv r \pmod{p}$, $r \in [1, p - 2]$ and $t \equiv q - 1 \pmod{q}$.
- (6) $\tau^t I_{p+r}$, with $t \geq 1$ such that $t \equiv r \pmod{q}$, $r \in [1, q - 2]$ and $t \equiv p - 1 \pmod{p}$.

Proof. Let $Y \cong \tau^k I_j$, for $k \geq 0$ and $j \in [0, p + q - 1]$ and M be a regular A -module. Hence $\text{Hom}_A(Y, M) = 0$. On the other hand, from the Auslander-Reiten formula, we have

$$\text{Ext}_A^1(\tau^k I_j, M) \cong D \text{Hom}_A(\tau^{-1} M, \tau^k I_j) \cong D \text{Hom}_A(\tau^{-(k+1)} M, I_j).$$

It follows that $\text{Ext}_A^1(\tau^k I_j, M) = 0$ if and only if, $j \notin \text{supp } \tau^{-(k+1)} M$. Therefore

$$(F, G, \tau^k I_j) \text{ is a s.s. if and only if, } j \in (\text{Supp } \tau^{-(k+1)} \mathcal{F})' \cap (\text{Supp } \tau^{-(k+1)} \mathcal{G})'.$$

We consider several cases in order to find all the *s.s.* of the form (F, G, Y) with Y preinjective.

Case 1. Let $Y \cong I_j$ with $j \in [0, p + q - 1]$. Then, by Corollary 8, we have that $(\text{Supp } \tau^{-1} \mathcal{F})' \cap (\text{Supp } \tau^{-1} \mathcal{G})' = \{1\} \cap \{p\} = \emptyset$.

Case 2. Let $Y \cong \tau^t I_j$, with $t \geq 1$ such that $p \mid t$ and $q \mid t$. Then

$$(\text{Supp } \tau^{-(t+1)} \mathcal{F})' \cap (\text{Supp } \tau^{-(t+1)} \mathcal{G})' = (\text{Supp } \tau^{-1} \mathcal{F})' \cap (\text{Supp } \tau^{-1} \mathcal{G})' = \emptyset.$$

Case 3. Let $Y \cong \tau^t I_j$, with t such that $p \nmid t$ and $q \mid t$. Let $t \equiv r \pmod{p}$, $r \in [1, p - 1]$. Here we need to consider two cases:

- If $r = p - 1$ then, using Corollary 8, we have that

$$\text{Supp } \tau^{-(t+1)} \mathcal{F} = \text{Supp } \tau^{-(r+1)} \mathcal{F} = \text{Supp } \tau^{-p} \mathcal{F} = \text{Supp } \mathcal{F} \text{ and}$$

$$\text{Supp } \tau^{-(t+1)} \mathcal{G} = \text{Supp } \tau^{-1} \mathcal{G} = [0, p + q - 1] \setminus \{p\}.$$

Hence $(\text{Supp } \tau^{-(t+1)} \mathcal{F})' \cap (\text{Supp } \tau^{-(t+1)} \mathcal{G})' = \{p\}$ and therefore $(F, G, \tau^t I_p)$ is a *s.s.*

- If $r \neq p - 1$, $\text{Supp } \tau^{-(t+1)} \mathcal{F} = \text{Supp } \tau^{-(r+1)} \mathcal{F} = [0, p + q - 1] \setminus \{r + 1\}$.
Then $(\text{Supp } \tau^{-(t+1)} \mathcal{F})' \cap (\text{Supp } \tau^{-(t+1)} \mathcal{G})' = \{r + 1\} \cap \{p\} = \emptyset$.

Case 4. Let $Y \cong \tau^t I_j$ with $t \geq 1$ such that $q \nmid t$ and $p \mid t$. Let $r \equiv t \pmod{q}$, with $r \in [1, q - 1]$. Consider two cases $r = q - 1$ and $r \neq q - 1$.

- If $r = q - 1$, then $\text{Supp } \tau^{-(t+1)} \mathcal{G} = \text{Supp } \tau^{-q} \mathcal{G} = \text{Supp } \mathcal{G}$ and $\text{Supp } \tau^{-(t+1)} \mathcal{F} = \text{Supp } \tau^{-1} \mathcal{F} = [0, p + q - 1] \setminus \{1\}$. It follows that $(\text{Supp } \tau^{-(t+1)} \mathcal{F})' \cap (\text{Supp } \tau^{-(t+1)} \mathcal{G})' = \{1\}$. Therefore $(F, G, \tau^t I_1)$ is a *s.s.*
- If $r \neq q - 1$, so we have that

$$\text{Supp } \tau^{-(t+1)} \mathcal{G} = \text{Supp } \tau^{-(r+1)} \mathcal{G} = [0, p + q - 1] \setminus \{p + r\}.$$

Then $(\text{Supp } \tau^{-(t+1)} \mathcal{F})' \cap (\text{Supp } \tau^{-(t+1)} \mathcal{G})' = \{1\} \cap \{p + r\} = \emptyset$.

Case 5. Let $Y \cong \tau^t I_j$, with t such that $q \nmid t$ and $p \nmid t$. Let r_1 and r_2 be such that $t \equiv r_1 \pmod{p}$, $t \equiv r_2 \pmod{q}$, $r_1 \in [1, p - 1]$ and $r_2 \in [1, q - 1]$. We will consider several cases:

- If $r_1 = p - 1$ and $r_2 = q - 1$, then we have:

$$(\text{Supp } \tau^{-(t+1)} \mathcal{F})' \cap (\text{Supp } \tau^{-(t+1)} \mathcal{G})' = (\text{Supp } \mathcal{F})' \cap (\text{Supp } \mathcal{G})' = \{0, p + q - 1\}.$$

Therefore $(F, G, \tau^t I_0)$ and $(F, G, \tau^t I_{p+q-1})$ are *s.s.*

- If $r_2 = q - 1$ and $r_1 \neq p - 1$. Then $\text{Supp } \tau^{-(t+1)} \mathcal{G} = \text{Supp } \mathcal{G}$ and

$$\text{Supp } \tau^{-(t+1)} \mathcal{F} = \text{Supp } \tau^{-(r_1+1)} \mathcal{F} = [0, p + q - 1] \setminus \{r_1 + 1\}.$$

Hence $(F, G, \tau^t I_{r_1+1})$ is a *s.s.*

- If $r_1 = p - 1$ and $r_2 \neq q - 1$, then $\text{Supp } \tau^{-(t+1)} \mathcal{F} = \text{Supp } \mathcal{F}$ and

$$\text{Supp } \tau^{-(t+1)} \mathcal{G} = \text{Supp } \tau^{-(r_2+1)} \mathcal{G} = [0, p + q - 1] \setminus \{p + r_2\}.$$

Then $(F, G, \tau^t I_{p+r_2})$ is a *s.s.*

- If $r_1 \neq p - 1$ and $r_2 \neq q - 1$, then

$$\text{Supp } \tau^{-(t+1)} \mathcal{G} = \text{Supp } \tau^{-(r_2+1)} \mathcal{G} = [0, p + q - 1] \setminus \{p + r_2\} \text{ and}$$

$$\text{Supp } \tau^{-(t+1)} \mathcal{F} = \text{Supp } \tau^{-(r_1+1)} \mathcal{F} = [0, p + q - 1] \setminus \{r_1 + 1\}.$$

Consequently, there is no *s.s.* in this case.

□

The following result is well-known and it is not difficult to prove.

Lemma 12. *Given a finite dimensional K -algebra B and*

$$0 \longrightarrow L \xrightarrow{(f' \ g')^t} X \oplus Y \xrightarrow{(f \ g)} M \longrightarrow 0$$

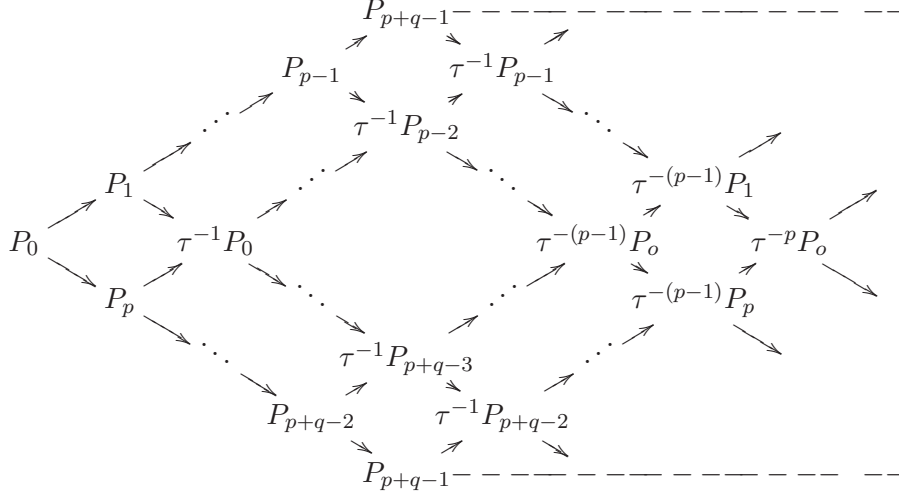
an almost split sequence in $\text{mod } B$. Then

- (1) *If f' is a monomorphism (resp. epimorphism), then g is a monomorphism (resp. epimorphism).*
- (2) *If g' is a monomorphism (resp. epimorphism), then f is a monomorphism (resp. epimorphism).*

Proposition 13. *Let $A = K(\widetilde{\mathbb{A}}_{p,q})$. The postprojective component $\mathcal{P}(A)$ of $\Gamma(\text{mod } A)$ has the following properties:*

- (1) *Any irreducible morphism $W \longrightarrow V$ between indecomposable modules in $\mathcal{P}(A)$ is a monomorphism.*
- (2) *The module P_{p+q-1} and its successors in $\mathcal{P}(A)$ are sincere.*
- (3) *If $0 < r < p$ then there is a projective P_i , for some $i \in [0, p + q + 1]$ such that $\tau^{-r}P_i$ is not sincere, moreover $\tau^{-t}P_i$ is sincere for all $t \geq p$ and $i \in [0, p + q + 1]$.*
- (4) *If $i \in [0, p - 1]$, then*
 - *The smallest integer r such that $\tau^{-r}P_i$ is sincere is $r = p - i$. Furthermore, all the modules of the form $\tau^{-k}P_i$, with $k > p - i$, are sincere.*
 - *If $k \in [0, p - i]$, then the composition factors of $\tau^{-k}P_i$ are S_j with $j \in [0, i + k] \cup [p, p + k - 1]$.*
 - *The composition factors of P_i are S_j with $j \in [0, i]$.*
- (5) *If $i \in [p, p + q - 2]$ and $p + q > 2$ then*
 - *If $i \leq q - 1$, then the smallest integer r such that $\tau^{-r}P_i$ is sincere is $r = p$. Furthermore, all the modules $\tau^{-k}P_i$, with $k > p$, are sincere.*
 - *If $i \in [q - 1, p + q - 1]$, then the smallest integer r such that $\tau^{-r}P_i$ is sincere is $r = p + q - 1 - i$. Furthermore, all the modules $\tau^{-k}P_i$, with $k > p + q - 1 - i$, are sincere.*
 - *If $\tau^{-k}P_i$ is not sincere, then its composition factors are S_j with $j \in [p, i + k] \cup [0, k]$.*
- (6) *If $k \in [1, p - 1]$, then the composition factors of $\tau^{-k}P_0$ are S_j with $j \in [0, k] \cup [p, p - 1 + k]$.*

Proof. Our proof uses the structure of the postprojective component $\mathcal{P}(A)$ of $\Gamma(\text{mod } A)$ which looks as follows:



In order to prove (1) we observe that in the almost split sequence

$$0 \longrightarrow P_0 \xrightarrow{(\alpha' \ \beta')^t} P_1 \oplus P_p \xrightarrow{(\alpha \ \beta)} \tau^{-1}P_0 \longrightarrow 0,$$

the morphisms α' and β' are mono, because are irreducible morphisms between projective modules. Then, by Lemma 12, α and β are mono too. Analogously, in all almost split sequences beginning in a projective module the arrows represent monomorphisms. Hence, according to the shape $\mathcal{P}(A)$, all the arrows in this component represent monomorphisms.

Now we prove (2). We note that in $\mathcal{P}(A)$ there are paths

$$P_0 \rightarrow P_1 \rightarrow \cdots \rightarrow P_{p-1} \rightarrow P_{p+q-1} \text{ and } P_0 \rightarrow P_p \rightarrow \cdots \rightarrow P_{p+q-1} \rightarrow P_{p+q-1},$$

where, by (1), all the arrows represent monomorphisms. Therefore, P_{p+q-1} and its successors are sincere modules.

For (3), we observe that none of the predecessors in $\mathcal{P}(A)$ of modules in the paths below

$$P_{p+q-1} \longrightarrow \tau^{-1}P_{p-1} \longrightarrow \tau^{-2}P_{p-2} \longrightarrow \cdots \longrightarrow \tau^{-(p-1)}P_0 \longrightarrow \tau^{-p}P_0 \quad (3.1)$$

$$P_{p+q-1} \longrightarrow \tau^{-1}P_{p+q-2} \longrightarrow \tau^{-2}P_{p+q-3} \longrightarrow \cdots \longrightarrow \tau^{-(p-1)}P_p \longrightarrow \tau^{-p}P_0 \quad (3.2)$$

have S_{p+q-1} as a composition factor. In particular, $\tau^{-(p-1)}P_0$ does not have S_{p+q-1} as a composition factor. Therefore, by (1), none of its predecessors have S_{p+q-1} as a composition factor. On the other hand, any successor of $\tau^{-(p-1)}P_0$, and any of modules in (3.1) and (3.2), are sincere.

In order to show (4), we note that all the modules $\tau^{-(p-i)}P_i$, with $i \in [0, p-1]$, are in the path (3.1) and therefore are sincere. Moreover, as stated earlier, no predecessor of a module which is in the path (3.1) has S_{p+q-1} as a

composition factor. It follows that, if $i \in [0, p-1]$ then none of the modules of the form $\tau^{-k}P_i$, with $k < p-i$, are sincere.

On the other hand, the predecessors of P_i , with $i \in [0, p-1]$, are the projectives P_0, P_1, \dots, P_{i-1} . Thus the composition factors of P_i , with $i \in [0, p-1]$, are S_j with $j \in [0, i]$. Analogously, for $i \in [p, p+q-2]$, the composition factors of P_i are S_j , with $j \in [p, i] \cup \{0\}$.

Let $i \in [0, p-1]$. Then in $\mathcal{P}(A)$ there are paths of the form

$$P_{i+k} \rightarrow \tau^{-1}P_{i+k-1} \rightarrow \tau^{-2}P_{i+k-2} \rightarrow \dots \rightarrow \tau^{-(k-1)}P_{i+1} \rightarrow \tau^{-k}P_i \quad (3.3)$$

and

$$P_{p+k-1} \rightarrow \tau^{-1}P_{p+k-2} \rightarrow \dots \rightarrow \tau^{-(k-1)}P_p \rightarrow \tau^{-k}P_0 \rightarrow \tau^{-k}P_1 \rightarrow \dots \rightarrow \tau^{-k}P_i \quad (3.4)$$

Furthermore, there are no paths between P_j , with $j \geq i+k$, and $\tau^{-k}P_i$. Hence the composition factors of $\tau^{-k}P_i$, up to multiplicity, are the composition factors of P_{i+k} and of P_{p+k-1} . In other words, the composition factors of $\tau^{-k}P_i$ are S_j with $j \in [0, i]$ and $i \in [p, p+q-2]$.

The proof of (5), is analogous to the previous and part (6) is left to the reader, since it follows by a similar argument. \square

Proposition 14. *Let $A = K(\widetilde{\mathbb{A}}_{p,q})$. The complete list of s.s. (M, F, G, Y) , where Y is a postprojective A -module, is as follows:*

- (1) (S_{p+q-1}, F, G, P_0) .
- (2) (M, F, G, P_{p+q-1}) , where

$$M \cong \begin{cases} \tau^{-p+1}P_0, & \text{if } p = q \\ \tau^{-p+1}P_{q-1}, & \text{if } p \neq q \end{cases}$$

- (3) $(M, F, G, \tau^{-t}P_{p+q-1})$ with $t \in \mathbb{N}$ such that $p \mid t$ and $q \mid t$, where

$$M \cong \begin{cases} \tau^{-t-p+1}P_0, & \text{if } p = q \\ \tau^{-t-p+1}P_{q-1}, & \text{if } p \neq q \end{cases}$$

- (4) $(\tau^{-t+1}P_{p+q-1}, F, G, \tau^{-t}P_0)$, with $t \in \mathbb{N}$ such that $p \mid t$ and $q \mid t$.
- (5) $(\tau^{-t-(p-r-1)}P_{q+r-1}, F, G, \tau^{-t}P_{p-r})$, with $t, r \in \mathbb{N}$ such that $q \mid t$, $t \equiv r \pmod{p}$ and $r \in [1, p-1]$.
- (6) $(M, F, G, \tau^{-t}P_{p+q-r-1})$, with $t, r \in \mathbb{N}$ such that $p \mid t$, $t \equiv r \pmod{q}$ and $r \in [1, q-1]$, where

$$M \cong \begin{cases} \tau^{-t-q+r+1}P_{p-q+r}, & \text{if } p \geq q-r \\ \tau^{-t-p+1}P_{q-r-1}, & \text{if } p < q-r \end{cases}$$

Proof. Proposition 10 characterizes the s.s. of the form (F, G, Y) , with Y a postprojective A -module. Using this characterization we obtain stratifying systems of type (X, F, G, Y) .

The quiver $\widetilde{\mathbb{A}}_{p,q}$ has $p+q$ vertices then, by Lemma 3 and Lemma 4, there is a unique module M such that $(M, F, G, \tau^{-t}P_l)$ is a c.s.s. Thus, the proof consists in the verification that the indecomposable module M satisfies the following conditions:

- $\text{Ext}_A^1(M, M) = 0$.
- $\text{Hom}_A(\mathcal{F}, M) = 0$ and $\text{Hom}_A(\mathcal{G}, M) = 0$.
- $\text{Ext}_A^1(\mathcal{F}, M) = 0$ and $\text{Ext}_A^1(\mathcal{G}, M) = 0$.
- $\text{Hom}_A(\tau^{-t}P_l, M) = 0$ and $\text{Ext}_A^1(\tau^{-t}P_l, M) = 0$.

We observe that all the modules M considered in this proof, except I_{p+q-1} , are of the form $M \cong \tau^{-k}P_j$, with $k \geq 0$, and therefore are indecomposable and have no self-extensions. Moreover, there is no nonzero morphism from a regular module to a posprojective module, that is, if R is a regular module then $\text{Hom}_A(R, \tau^{-k}P_j) = 0$. On the other hand, by the Auslander-Reiten formula, we have that

$$\text{Ext}_A^1(R, \tau^{-k}P_j) \cong D \text{Hom}_A(\tau^{-k}P_j, \tau R) \cong D \text{Hom}_A(P_j, \tau^{k+1}R).$$

Therefore we have that $\text{Ext}_A^1(\mathcal{F}, \tau^{-k}P_j) = 0$ and $\text{Ext}_A^1(\mathcal{G}, \tau^{-k}P_j) = 0$ if and only if, $j \in (\text{Supp } \tau^{k+1}\mathcal{F})' \cap (\text{Supp } \tau^{k+1}\mathcal{G})'$. In view of these observations, if $(F, G, \tau^{-t}P_l)$ is a *s.s.* and $X \cong \tau^{-k}P_j$, in order to show that $(X, F, G, \tau^{-t}P_l)$ is a *s.s.* it is sufficient to check the following conditions:

- $j \in (\text{Supp } \tau^{k+1}\mathcal{F})' \cap (\text{Supp } \tau^{k+1}\mathcal{G})'$.
- $\text{Hom}_A(\tau^{-t}P_l, M) = 0$ (or equivalently $[\tau^t M : S_l] = 0$).
- $\text{Ext}_A^1(\tau^{-t}P_l, M) = 0$.

In what follows we find these conditions for each of the sequences stated in the proposition.

(1) First let us check that (S_{p+q-1}, F, G, P_0) is a *s.s.* As the vertex $p+q-1$ is a source then the $S_{p+q-1} \cong I_{p+q-1}$ and thus

$$\text{Ext}_A^1(\mathcal{F}, S_{p+q-1}) = 0, \text{Ext}_A^1(\mathcal{G}, S_{p+q-1}) = 0 \text{ and } \text{Ext}_A^1(P_0, S_{p+q-1}) = 0.$$

On the other hand, since $p+q-1 \notin (\text{Supp } \mathcal{F} \cup \text{Supp } \mathcal{G})$ then $\text{Hom}_A(\mathcal{F}, S_{p+q-1}) = 0$, $\text{Hom}_A(\mathcal{G}, S_{p+q-1}) = 0$ and $\text{Hom}_A(P_0, S_{p+q-1}) = 0$.

(2) To complete the *s.s.* (F, G, P_{p+q-1}) we have to consider two cases:

- If $p = q$ and $M \cong \tau^{-p+1}P_0$, then by Corollary 8 we have that $(\text{Supp } \tau^p\mathcal{F})' \cap (\text{Supp } \tau^p\mathcal{G})' = \{0, p+q-1\}$. On the other hand $[\tau^{-p+1}P_0 : S_{p+q-1}] = 0$, by Proposition 13 (6). Therefore $(\tau^{-p+1}P_0, F, G, P_{p+q-1})$ is a *c.s.s.*
- If $p \neq q$ and $M \cong \tau^{-p+1}P_{q-1}$ then

$$(\text{Supp } \tau^p\mathcal{F})' \cap (\text{Supp } \tau^p\mathcal{G})' = \{q-1\}$$

and, by Proposition 13 (5), we have that $[\tau^{-p+1}P_{q-1} : S_{p+q-1}] = 0$.

(3) In order to complete the *s.s.* $(F, G, \tau^{-t}P_{p+q-1})$, with $t \geq 1$ such that $p \mid t$ and $q \mid t$, we consider the following two cases:

- Suppose $p = q$. We claim that $(\tau^{-t-p+1}P_0, F, G, \tau^{-t}P_{p+q-1})$, is a *c.s.s.* In fact, $(\text{Supp } \tau^{t+p}\mathcal{F})' \cap (\text{Supp } \tau^{t+p}\mathcal{G})' = \{0, p+q-1\}$. On the other hand, by Proposition 13 (6), $\text{Hom}_A(\tau^{-t}P_{p+q-1}, \tau^{-t-p+1}P_0) \cong 0$. Moreover, by Lemma 9, $\text{Ext}_A^1(\tau^{-t}P_0, \tau^{-t-p+1}P_{q-1}) = 0$.

- Suppose $p \neq q$. In this case $(\tau^{-t-p+1}P_{q-1}, F, G, \tau^{-t}P_{p+q-1})$ is a *c.s.s.*. Indeed, $(\text{Supp } \tau^{t+p}\mathcal{F})' \cap (\text{Supp } \tau^{t+p}\mathcal{G})' = \{q-1\}$, by Corollary 8, and $[\tau^{-p+1}P_{q-1} : S_{p+q-1}] = 0$, by Proposition 13 (5). Finally, by Lemma 9, we have that $\text{Ext}_A^1(\tau^{-t}P_{p+q-1}, \tau^{-t-p+1}P_{q-1}) = 0$.

(4) We show that $(\tau^{-t+1}P_{p+q-1}, F, G, \tau^{-t}P_0)$, with $t \geq 1$ such that $p \mid t$ and $q \mid t$ is a *c.s.s.* In fact, $(\text{Supp } \tau^t\mathcal{F})' \cap (\text{Supp } \tau^t\mathcal{G})' = \{0, p+q-1\}$. And by the Auslander-Reiten formula we have

$$\begin{aligned} \text{Hom}_A(\tau^{-t}P_0, \tau^{-t+1}P_{p+q-1}) &\cong D \text{Ext}_A^1(P_{p+q-1}, \tau^{-1}P_0) \cong 0, \\ \text{Ext}_A^1(\tau^{-t}P_0, \tau^{-t+1}P_{p+q-1}) &\cong D \text{Hom}_A(P_{p+q-1}, P_0) \cong 0. \end{aligned}$$

Then the conditions for the *s.s.* are verified.

(5) We prove that $(\tau^{-t-(p-r-1)}P_{q+r-1}, F, G, \tau^{-t}P_{p-r})$, with $t, r \in \mathbb{N}$ such that $q \mid t$, $t \equiv r \pmod{p}$ and $r \in [1, p-1]$ is a *s.s.* According to Corollary 8, we have

$$(\text{Supp } \tau^{t+(p-r)}\mathcal{F})' \cap (\text{Supp } \tau^{t+(p-r)}\mathcal{G})' = (\text{Supp } \mathcal{F})' \cap (\text{Supp } \tau^{p-r}\mathcal{G})' = \{q+r-1\}.$$

Applying Lemma 9 we can assert that

$$\text{Ext}_A^1(\tau^{-t}P_{p-r}, \tau^{-t-(p-r-1)}P_{q+r-1}) = 0.$$

And finally, by Proposition 13 (5), we have

$$\text{Hom}_A(\tau^{-t}P_{p-r}, \tau^{-t-(p-r-1)}P_{q+r-1}) \cong \text{Hom}_A(P_{p-r}, \tau^{-(p-r-1)}P_{q+r-1}) \cong 0.$$

(6) Suppose that $t, r \in \mathbb{N}$ are such that $p \mid t$, $t \equiv r \pmod{q}$ and $r \in [1, q-1]$. Let us consider two cases:

- Assume that $p \geq q-r$. Then, by Corollary 8, we have that

$$\begin{aligned} (\text{Supp } \tau^{t+q-r}\mathcal{F})' \cap (\text{Supp } \tau^{t+q-r}\mathcal{G})' &= (\text{Supp } \tau^{q-r}\mathcal{F})' \cap (\text{Supp } \mathcal{G})' \\ &= \{p-q+r\}. \end{aligned}$$

We note that $q-r-1 \geq 0$, then by Lemma 9 we have that

$$\text{Ext}_A^1(\tau^{-t}P_{p+q-r-1}, \tau^{-t-q+r+1}P_{p-q+r}) = 0.$$

Furthermore, by Proposition 13 (4), it follows that

$$\begin{aligned} \text{Hom}_A(\tau^{-t}P_{p+q-r-1}, \tau^{-t-q+r+1}P_{p-q+r}) &\cong \text{Hom}_A(P_{p+q-r-1}, \tau^{-q+r+1}P_{p-q+r}) \\ &\cong 0. \end{aligned}$$

Thus $(\tau^{-t-q+r+1}P_{p-q+r}, F, G, \tau^{-t}P_{p+q-r-1})$ is a *c.s.s.*

- Suppose that $p < q-r$. Let l such that $p+l = q-r$. Therefore, by Corollary 8, we have that

$$\begin{aligned} (\text{Supp } \tau^{t+p}\mathcal{F})' \cap (\text{Supp } \tau^{t+p}\mathcal{G})' &= (\text{Supp } \mathcal{F})' \cap (\text{Supp } \tau^{-l}\mathcal{G})' \\ &= \{p+l-1\} \\ &= \{q-r-1\}. \end{aligned}$$

Now, by Lemma 9, $\text{Ext}_A^1(\tau^{-t}P_{p+q-r-1}, \tau^{-t-p+1}P_{q-r-1}) = 0$. Finally, by Proposition 13 (4), we have that $S_{p+q-r-1}$ is not a composition factor of $\tau^{-p+1}P_{q-r-1}$.

□

Next result is proved in an analogous way as Proposition 13.

Proposition 15. *Let $A = K(\tilde{\mathbb{A}}_{p,q})$. The preinjective component $\mathcal{Q}(A)$ of $\Gamma(\text{mod } A)$ has the following properties:*

- (1) *Any irreducible morphism $W \rightarrow V$ between indecomposable modules in $\mathcal{Q}(A)$ is an epimorphism.*
- (2) *The A -module I_0 and its predecessors in $\mathcal{Q}(A)$ are sincere.*
- (3) *If $0 < r < p$ then there is an injective I_i , for some $i \in [0, p + q + 1]$ such that $\tau^r I_i$ is not sincere, moreover $\tau^t I_i$ is sincere for all $t \geq p$ and $i \in [0, p + q + 1]$.*
- (4) *If $i \in [1, p - 1]$, then:*
 - *the smallest integer r such that $\tau^r I_i$ is a sincere module is $r = i$. Furthermore, all the modules of the form $\tau^k I_i$, with $k \geq i$, are sincere.*
 - *if $k < i$, then the composition factors of $\tau^k I_i$ are simple S_j with $j \in [i - k, p - 1] \cup [p + q - k, p + q - 1]$.*
- (5) *If $p \leq i < p + q - 1$, then:*
 - *if $p \leq i < q - 1$, the smallest integer r such that $\tau^r I_i$ is a sincere module is $r = i - p + 1$. Moreover, all the modules $\tau^r I_i$, with $r \geq i - p + 1$, are sincere.*
 - *if $i \in [q - 1, p + q - 1]$, the smallest integer r such that $\tau^r I_i$ is a sincere A -module is $r = p$. Furthermore, all the modules $\tau^r I_i$, with $r \geq p$ are sincere.*
 - *if $i \in [p, p + q - 2]$ and k is such that $\tau^k I_i$ is not a sincere module, then the composition factors of $\tau^k I_i$ are simple modules S_j with $j \in [i - k, p + q - 1] \cup [p - k, p - 1]$.*
- (6) *If $k < p$, then the composition of $\tau^k I_{p+q-1}$ are simple modules S_j with $j \in [p - k, p - 1] \cup [p + q - k, p + q - 1]$.*

Proposition 16. *Let $A = K(\tilde{\mathbb{A}}_{p,q})$. The complete list of c.s.s. (M, F, G, Y) , where Y is a preinjective A -module is the following:*

- (1) $(\tau^t I_{p-1}, F, G, \tau^t I_p)$, with $t \geq 1$ such that $t \equiv p - 1 \pmod{p}$ and $q|t$.
- (2) $(\tau^t I_{p+q-2}, F, G, \tau^t I_1)$, with $t \geq 1$ such that $t \equiv q - 1 \pmod{q}$ and $p|t$.
- (3) $(\tau^{t+1} I_{p+q-1}, F, G, \tau^t I_0)$, with $t \geq 1$ such that $t \equiv p - 1 \pmod{p}$ and $t \equiv q - 1 \pmod{q}$.
- (4) $(\tau^{t-p+1} I_{q-1}, F, G, \tau^t I_{p+q-1})$, with $t \geq 1$ such that $t \equiv p - 1 \pmod{p}$ and $t \equiv q - 1 \pmod{q}$.
- (5) $(\tau^{t-r} I_{p+q-r-2}, F, G, \tau^t I_{r+1})$, with $t \geq 1$ such that $t \equiv r \pmod{p}$, $r \in [1, p - 2]$ and $t \equiv q - 1 \pmod{q}$.
- (6) $(M, F, G, \tau^t I_{p+r})$, with $t \geq 1$ such that $t \equiv r \pmod{q}$, $r \in [1, q - 2]$ and $t \equiv p - 1 \pmod{p}$, where

$$M \cong \begin{cases} \tau^{t-r} I_{p-(r+1)}, & \text{if } r < p \\ \tau^{t-(p-1)} I_r, & \text{if } r \geq p \end{cases}$$

Proof. By Lemma 3 and Lemma 4 there is a unique module M such that $(M, F, G, \tau^m I_i)$ is a *s.s.* Therefore the proof consists in verify that the indecomposable module M satisfies the following conditions:

- $\text{Ext}_A^1(M, M) = 0$.
- $\text{Hom}_A(\mathcal{F}, M) = 0$ and $\text{Hom}_A(\mathcal{G}, M) = 0$.
- $\text{Ext}_A^1(\mathcal{F}, X) = 0$ and $\text{Ext}_A^1(\mathcal{G}, M) = 0$.
- $\text{Hom}_A(\tau^t I_i, M) = 0$ and $\text{Ext}_A^1(\tau^t I_i, M) = 0$.

All modules M that we will consider are of the form $M \cong \tau^l I_j$ and therefore are indecomposable and have no self-extensions. Moreover if M is a preinjective and R is a regular then, using the Auslander-Reiten formula, we have $\text{Ext}_A^1(R, M) \cong D \text{Hom}_A(M, \tau R) \cong 0$. On the other hand, for $l \geq 0$, we have $\text{Hom}_A(R, \tau^l I_j) \cong \text{Hom}_A(\tau^{-l} R, I_j)$. It follows that $\text{Hom}_A(R, \tau^l I_j) = 0$ if and only if $j \notin \text{supp } \tau^{-l} R$. Therefore we have that $\text{Hom}_A(\mathcal{F}, \tau^l I_j) = 0$ and $\text{Hom}_A(\mathcal{G}, \tau^l I_j) = 0$ if and only if $j \in (\text{Supp } \tau^{-l} \mathcal{F})' \cap (\text{Supp } \tau^{-l} \mathcal{G})'$.

According to the above remarks, if $(F, G, \tau^t I_i)$ is a *s.s.* and $M \cong \tau^l I_j$, in order to prove that $(M, F, G, \tau^t I_i)$ is a *c.s.s.* it is sufficient to show the following conditions: $j \in (\text{Supp } \tau^{-l} \mathcal{F})' \cap (\text{Supp } \tau^{-l} \mathcal{G})'$, $\text{Hom}_A(\tau^t I_i, M) = 0$ and $\text{Ext}_A^1(\tau^t I_i, M) = 0$.

We consider various possibilities, which depend on the form of t , according to the list of the statement of Proposition 11.

(1) Let $t \geq 1$ such that $t \equiv p - 1 \pmod{p}$ and $q \mid t$. Then, by Corollary 8, we have

$$(\text{Supp } \tau^{-t} \mathcal{F})' \cap (\text{Supp } \tau^{-t} \mathcal{G})' = (\text{Supp } \tau^{-(p-1)} \mathcal{F})' \cap (\text{Supp } \tau^{-t} \mathcal{G})' = \{p - 1\}.$$

We observe that $\text{Hom}_A(\tau^t I_p, \tau^t I_{p-1}) \cong \text{Hom}_A(I_p, I_{p-1}) \cong 0$, because I_p has no S_{p-1} as a composition factor.

Moreover, by Lemma 9, $\text{Ext}_A^1(\tau^t I_p, \tau^t I_{p-1}) = 0$. Then $(\tau^t I_{p-1}, F, G, \tau^t I_p)$ is a *s.s.*

(2) Let $t \geq 1$ such that $t \equiv q - 1 \pmod{q}$ and $p \mid t$. By Corollary 8 we have that

$$(\text{Supp } \tau^{-t} \mathcal{F})' \cap (\text{Supp } \tau^{-t} \mathcal{G})' = (\text{Supp } \tau^{-t} \mathcal{F})' \cap (\text{Supp } \tau^{-(q-1)} \mathcal{G})' = \{p + q - 2\}.$$

On the other hand, since S_{p+q-2} is not a composition factor of I_1 we see that $\text{Hom}_A(\tau^t I_1, \tau^t I_{p+q-2}) \cong \text{Hom}_A(I_1, I_{p+q-2}) \cong 0$.

Finally, by Lemma 9, it follows that $\text{Ext}_A^1(\tau^t I_1, \tau^t I_{p+q-2}) = 0$.

(3) Let $t \geq 1$ such that $t \equiv p - 1 \pmod{p}$ and $t \equiv q - 1 \pmod{q}$. By Corollary 8 we have that

$$(\text{Supp } \tau^{-t-1} \mathcal{F})' \cap (\text{Supp } \tau^{-t-1} \mathcal{G})' = (\text{Supp } \mathcal{F})' \cap (\text{Supp } \mathcal{G})' = \{0, p + q - 1\}.$$

From the Auslander-Reiten formula it may be concluded that

$$\text{Hom}_A(\tau^t I_0, \tau^{t+1} I_{p+q-1}) \cong \text{Hom}_A(I_0, \tau I_{p+q-1}) \cong D \text{Ext}_A^1(I_{p+q-1}, I_0) = 0 \text{ and}$$

$$\begin{aligned} \text{Ext}_A^1(\tau^t I_0, \tau^{t+1} I_{p+q-1}) &\cong D \text{Hom}_A(\tau^{t+1} I_{p+q-1}, \tau^{t+1} I_0) \\ &\cong \text{Hom}_A(I_{p+q-1}, I_0) \\ &= 0. \end{aligned}$$

(4) Let $t \geq 1$ such that $t \equiv p-1 \pmod{p}$ and $t \equiv q-1 \pmod{q}$. We show that $(\tau^{t-p+1}I_{q-1}, F, G, \tau^t I_{p+q-1})$ is a *c.s.s.* over A . By Corollary 8 it follows that

$$(\text{Supp } \tau^{-t+p-1}\mathcal{F})' \cap (\text{Supp } \tau^{-t+p-1}\mathcal{G})' = (\text{Supp } \mathcal{F})' \cap (\text{Supp } \tau^{-(q-p)}\mathcal{G})' = \{q-1\}.$$

Since, $\text{Hom}_A(\tau^t I_{p+q-1}, \tau^{t-(p-1)} I_{q-1}) \cong \text{Hom}_A(\tau^{p-1} I_{p+q-1}, I_{q-1}) = 0$, because of Proposition 15 (5) we get that $[\tau^{p-1} I_{p+q-1} : S_{q-1}] = 0$. Moreover, by Lemma 9, we have that $\text{Ext}_A^1(\tau^t I_{p+q-1}, \tau^{t-(p-1)} I_{q-1}) = 0$.

(5) Let $t \geq 1$ such that $t \equiv r \pmod{p}$, $r \in [1, p-2]$ and $t \equiv q-1 \pmod{q}$ and $M \cong \tau^{t-r} I_{p+q-r-2}$. First, by Corollary 8, we have that

$$\begin{aligned} (\text{Supp } \tau^{-(t-r)}\mathcal{F})' \cap (\text{Supp } \tau^{-(t-r)}\mathcal{G})' &= (\text{Supp } \mathcal{F})' \cap (\text{Supp } \tau^{-(t-r)}\mathcal{G})' \\ &= (\text{Supp } \tau^{-(q-1-r)}\mathcal{G})' \\ &= \{p+q-r-2\}. \end{aligned}$$

Furthermore, $\text{Hom}_A(\tau^t I_{r+1}, \tau^{t-r} I_{p+q-r-2}) \cong \text{Hom}_A(\tau^r I_{r+1}, I_{p+q-r-2}) = 0$, because, accordingly with Proposition 15 (5), $\tau^r I_{r+1}$ has no $S_{p+q-r-2}$ as a composition factor. Finally, by Lemma 9, we have that

$$\text{Ext}_A^1(\tau^t I_{r+1}, \tau^{t-r} I_{p+q-r-2}) = 0.$$

(6) Let $t \geq 1$ such that $t \equiv r \pmod{q}$, $r \in [1, q-2]$ and $t \equiv p-1 \pmod{p}$. To complete the *s.s.* $(F, G, \tau^t I_{p+r})$ we consider two situations:

- If $r < p$. Let $M \cong \tau^{t-r} I_{p-(r+1)}$. We have that

$$\begin{aligned} (\text{Supp } \tau^{-(t-r)}\mathcal{F})' \cap (\text{Supp } \tau^{-(t-r)}\mathcal{G})' &= (\text{Supp } \tau^{-(p-1-r)}\mathcal{F})' \cap (\text{Supp } \mathcal{G})' \\ &= (\text{Supp } \tau^{(r+1)}\mathcal{F})' \\ &= \{p-(r+1)\}. \end{aligned}$$

Since $[\tau^r I_{p+r} : S_{p-(r+1)}] = 0$, by Proposition 16 (5), then

$$\text{Hom}_A(\tau^t I_{p+r}, \tau^{t-r} I_{p-(r+1)}) \cong \text{Hom}_A(\tau^r I_{p+r}, I_{p-(r+1)}) = 0.$$

By Lemma 9, we have that $\text{Ext}_A^1(\tau^t I_{p+r}, \tau^{t-r} I_{p-(r+1)}) = 0$.

- If $r \geq p$. Let $M \cong \tau^{t-(p-1)} I_r$. Then we have that

$$\begin{aligned} (\text{Supp } \tau^{-[t-(p-1)]}\mathcal{F})' \cap (\text{Supp } \tau^{-[t-(p-1)]}\mathcal{G})' &= (\text{Supp } \tau^{-[r-(p-1)]}\mathcal{G})' \\ &= \{p+(r-p+1)-1\} \\ &= \{r\}. \end{aligned}$$

According to Proposition 16 (5), we have that

$$\text{Hom}_A(\tau^t I_{p+r}, \tau^{t-(p-1)} I_r) \cong \text{Hom}_A(\tau^{p-1} I_{p+r}, I_r) \cong 0$$

and by Lemma 9 we have that $\text{Ext}_A^1(\tau^t I_{p+r}, \tau^{t-(p-1)} I_r) = 0$.

□

Finally, Proposition 14 and Proposition 16 together are the main result of this paper, which establish the complete list of *c.s.s.* of the form (X, F, G, Y) .

Theorem 17. *Let $A = K(\tilde{\mathbb{A}}_{p,q})$. The complete list of *c.s.s.* of the form (M, F, G, Y) is as follows:*

(1) (S_{p+q-1}, F, G, P_0) .

(2) (M, F, G, P_{p+q-1}) , where

$$M \cong \begin{cases} \tau^{-p+1}P_0, & \text{if } p = q \\ \tau^{-p+1}P_{q-1}, & \text{if } p \neq q. \end{cases}$$

(3) $(M, F, G, \tau^{-t}P_{p+q-1})$ with $t \geq 1$ such that $p \mid t$ and $q \mid t$, where

$$M \cong \begin{cases} \tau^{-t-p+1}P_0, & \text{if } p = q \\ \tau^{-t-p+1}P_{q-1}, & \text{if } p \neq q. \end{cases}$$

(4) $(\tau^{-t+1}P_{p+q-1}, F, G, \tau^{-t}P_0)$, with $t \geq 1$ such that $p \mid t$ and $q \mid t$.

(5) $(\tau^{-t-(p-r-1)}P_{q+r-1}, F, G, \tau^{-t}P_{p-r})$, with $t \geq 1$ such that $q \mid t$, $t \equiv r \pmod{p}$ and $r \in [1, p-1]$.

(6) $(M, F, G, \tau^{-t}P_{p+q-r-1})$, with $t \geq 1$ such that $p \mid t$, $t \equiv r \pmod{q}$ and $r \in [1, q-1]$, where

$$M \cong \begin{cases} \tau^{-t-q+r+1}P_{p-q+r}, & \text{if } p \geq q-r \\ \tau^{-t-p+1}P_{q-r-1}, & \text{if } p < q-r. \end{cases}$$

(7) $(\tau^t I_{p-1}, F, G, \tau^t I_p)$, with $t \geq 1$ such that $t \equiv p-1 \pmod{p}$ and $q \mid t$.

(8) $(\tau^t I_{p+q-2}, F, G, \tau^t I_1)$, with $t \geq 1$ such that $t \equiv q-1 \pmod{q}$ and $p \mid t$.

(9) $(\tau^{t+1} I_{p+q-1}, F, G, \tau^t I_0)$, with $t \geq 1$ such that $t \equiv p-1 \pmod{p}$ and $t \equiv q-1 \pmod{q}$.

(10) $(\tau^{t-p+1} I_{q-1}, F, G, \tau^t I_{p+q-1})$, with $t \geq 1$ such that $t \equiv p-1 \pmod{p}$ and $t \equiv q-1 \pmod{q}$.

(11) $(\tau^{t-r} I_{p+q-r-2}, F, G, \tau^t I_{r+1})$, with $t \geq 1$ such that $t \equiv r \pmod{p}$, $r \in [1, p-2]$ and $t \equiv q-1 \pmod{q}$.

(12) $(M, F, G, \tau^t I_{p+r})$, with $t \geq 1$ such that $t \equiv r \pmod{q}$, $r \in [1, q-2]$ and $t \equiv p-1 \pmod{p}$, where

$$M \cong \begin{cases} \tau^{t-r} I_{p-(r+1)}, & \text{if } r < p \\ \tau^{t-(p-1)} I_r, & \text{if } r \geq p. \end{cases}$$

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